Mathematical and Musical Analysis of Kol Nidrei

Kol Nidrei, Op. 47, is an instrumental piece written in 1881 by Max Bruch. It was originally composed for violincello and orchestra, though the version that will be analyzed here is a translation for viola and piano.

Though Bruch was a protestant, he composed Kol Nidrei by drawing from two themes of Jewish descent. The first theme is featured in the opening melody and throughout the D minor section of the piece. It is based on the Kol Nidre prayer, which is an old Hebrew prayer of atonement that is recited on the Jewish holiday of Yom Kippur. The melody of this theme is meant to imitate the voice of a hazzan, who leads the worship of the congregation through lyrical prayer in a synagogue. The second theme is featured in the second half of the piece, where the music switches from D minor to D major. This theme draws from a section of a Hebrew melody titled "O Weep for Those that Wept on Babel's Stream" by Lord Byron.

Bruch composed Kol Nidrei in the romantic era of music. It was in this era that composers strayed from previous musical conventions and rules in order to create pieces consisting of significant emotional expression. Because of this, Kol Nidrei breaks many musical rules by utilizing irregular phrasing, disjunct harmonies, and little repetition in the melodic line. This lack of consistency makes it a difficult piece to examine, though spectrograms are able to help illustrate exactly what is heard in this piece.

Since Kol Nidrei has such emotional themes, the solo violist can enhance the emotion using vibrato. The use of vibrato in this case can be thought to imitate the human voice, especially that of a hazzan. In the recording, this vibrato is clearly heard when the first melody is introduced. This occurs at time 0:36 in the recording and at measure nine in the score, per Figure 1.2. However, in this recording the violist elected to perform measures 9-12 up one octave than as written, and perform measures 13-16 down one octave as written. Since these two sections are the same melodic line, the performer essentially interchanged them. Figure 1.1 below shows a spectrogram of this melodic section. Since the piano part consists only of quiet and steady chords, one can easily differentiate between the viola and piano parts in this spectrogram. The notes of the viola are not straight like those of the piano; they are very wavy, indicating the slight variance in the pitch as the violist uses vibrato. The use of vibrato can be seen in almost any portion of the spectrogram for this piece. In addition, at about time 0:43.5, the spectrogram shows a nearly vertical line linking the pitches of A_4 and $C\sharp_5$. This indicates that the performer slid his hand position during the slur that connects these two notes. This slur can be seen in the third beat of measure 10 and again in measure 14.
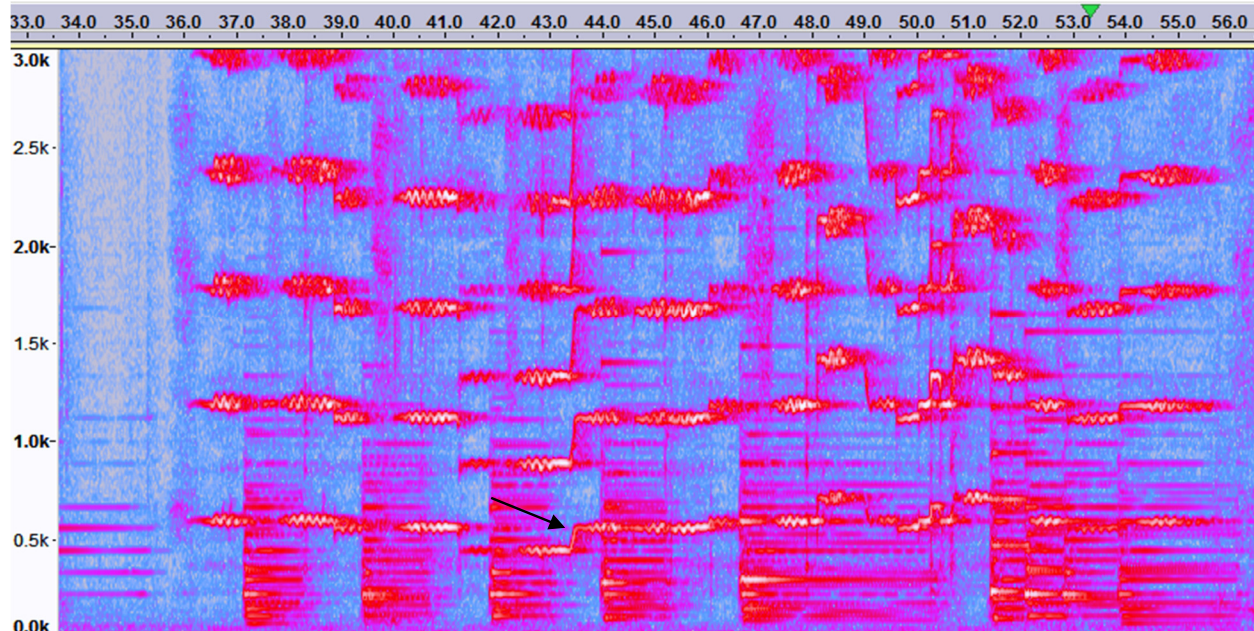
Figure 1.1: Spectrogram of measures 9-12. The arrow indicates the slur from A_4 to $C\sharp_5$.

Figure 1.2: Score excerpt for measures 9-12.

The next section that will be examined features a strong piano part the contrasts the sweet, lyrical sound of the viola. At time 2:17-2:26 and measures 29-30, the violist rests while the piano loudly plays three octaves simultaneously of the same pitch to a rigorous rhythm. The vertical lines at the beginning of every set of piano notes in Figure 2.1 below represent the percussive element of the piano: the striking of the hammers on the strings inside of the piano. This heavy, percussive sound is complemented from time 2:26-2:35 and measures 31-32 by a vibratoed, melodic line in the viola, which is then followed by a repeat of the piano part from time 2:17.

As seen in Figure 2.2, the first pitches of the piano section are C's (C_2, C_3, and C_4). These pitches have frequencies of 65, 131, and 262 Hz, respectively. In this section, the harmonics of these three pitches are clearly shown on the spectrogram in Figure 1.2. Examples of these occur at 392 Hz (G_4), 523 Hz (C_5), 659 Hz (E_5), 784 Hz (G_5), 1047 Hz (C_6), and 1319 Hz (E_6) as shown

below. The pitch at 262 Hz (C_4) is one of the pitches being played by the piano, though it is also a harmonic of the C_2 and C_3 being playing by the piano. This explains why the brightest line in the spectrogram occurs at 262 Hz.

Another way contrast is shown here is from measure 31-32. The chord on the second beat of measure 31 is consonant, since all the notes played in the viola and piano are part of a C major chord (C, E, G). The downbeat of measure 32 contrasts with this by being dissonant. This dissonance occurs because the lowest notes in the piano line remain the same, C_2 and C_3, while the viola and top line of the piano shift to a G major seven chord, which C is not a part of. The contrast of consonance versus dissonance can be seen in the spectrogram because the piano notes from measure 31 are straight, while the notes from measure 32 appear fuzzier.
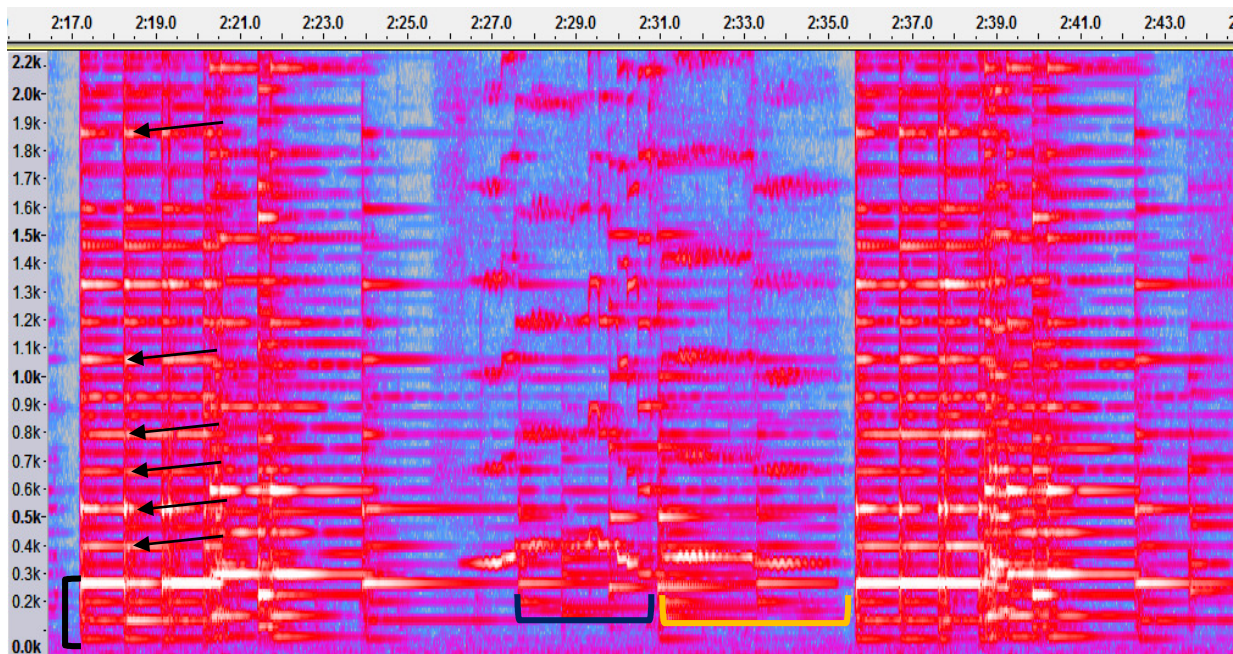

Figure 2.1: Spectrogram of measures 29-34. The black bracket indicates the three pitches being played in the piano. The arrows indicate examples of the harmonics of these three pitches. The blue bracket indicates the notes in measure 31, and the yellow bracket indicates the notes in measure 32.

Figure 2.2: Score excerpt of measures 29-36. The red rectangles indicate the consonant and dissonant chords described above.

In the middle of the minor section of this piece, there occurs a very high point in the music. This is apparent through the composer's use of sforzandos (a sudden emphasis on a particular note) and the abundance of fast moving notes in the viola part. This occurs from measures 41-46, and Figure 3.1 below shows the spectrogram of measures 41-43 from this section. As seen in Figure 3.2, each of these three measures contains a run of very fast notes either going up or down a portion of a scale. These runs are also slurred together. Figure 3.1 shows that these runs occur at time 3:14, 3:18, and 3:24.5. They make diagonal lines in the spectrogram, not individual horizontal lines that represent each individual note in each run. This is because the notes are played so quickly that the software picks it up almost as a glissando, which is also how the listener can perceive it.
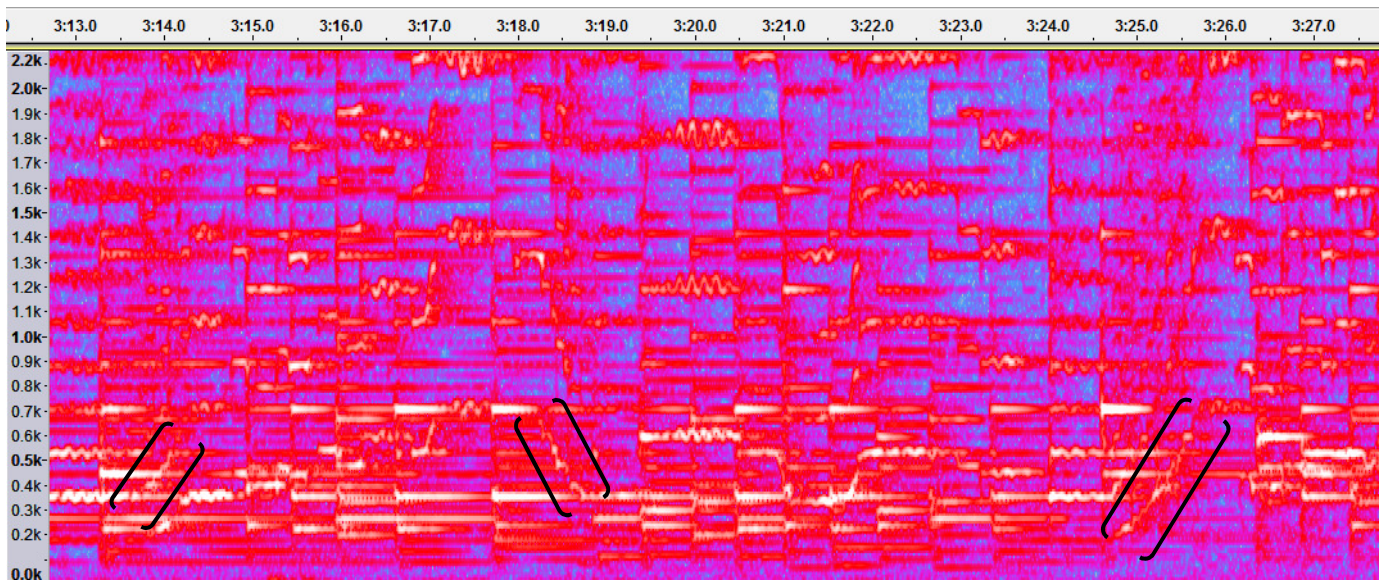
Figure 3.1: Spectrogram of measures 41-43. The bracketed portions show how the runs of notes are expressed visually.

Figure 3.2: Score excerpt of measures 41-13. The red circles indicate the three runs.

A musical technique that is used in Kol Nidrei a few different times is a trill. For instance, a trill is used in measure 46 to lead into measure 47. In Figure 4.1, this is shown to occur at time 3:38 in the music and last until 3:40.5. The trill alternates between frequencies of 330 and 349 Hz (E_4 and F_4), which can be seen with alternating horizontal lines. This alternation is exaggerated in the higher harmonics. The trill starts out slow, but speeds up as the lines become shorter in length. At the end of the trill, the score calls for a small turn, which means the violist must go down to a D_4 and back to an E_4 to finish. The performer in this recording elected to slow down at the turn in order to lead into the following measure. Therefore, the lines that represent D_4 and E_4 and much longer than those of the rest of the trill.
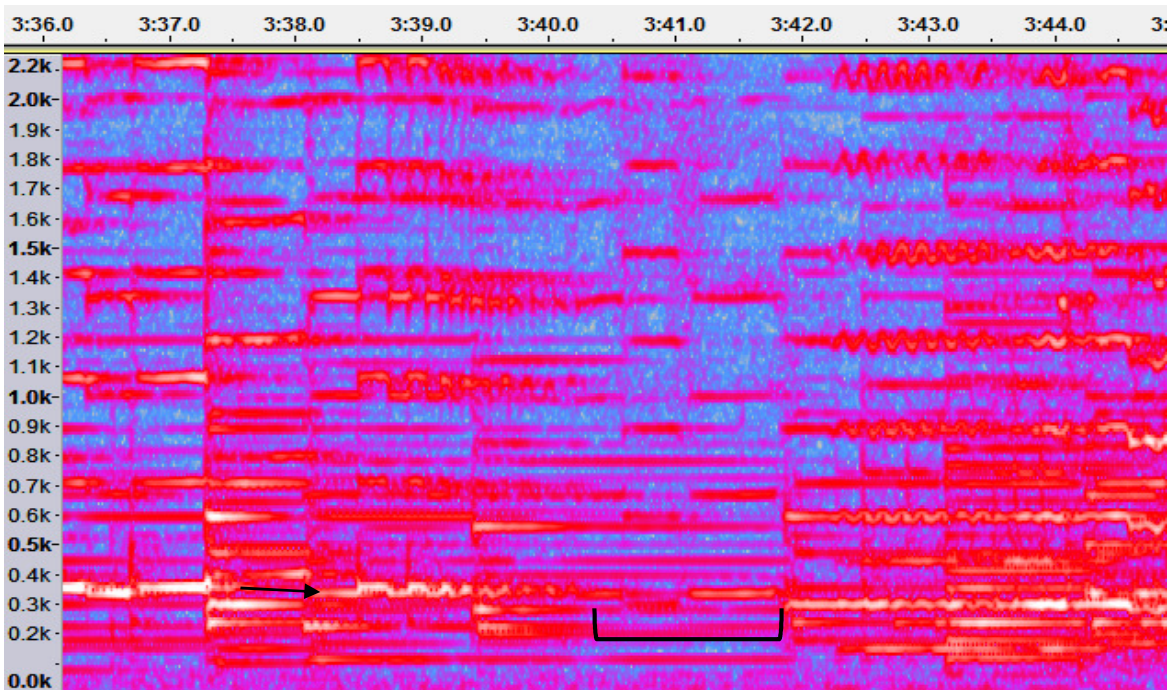
Figure 4.1: Spectrogram of measure 46 into 47. The arrow indicates the trill and the bracket indicates the turn.

Figure 4.2: Score excerpt of measures 46 and 47.

Another trill occurs in measure 89, or 6:47.5 in the recording. This trill takes up twice as much time as the previous one, as shown in the spectrogram in Figure 5.1 and the score excerpt in Figure 5.2. Just like the previous trill, this one starts out with a slow alternation, this time between frequencies of 587 and 659 Hz (D_5 and E_5). The horizontal lines which represent these frequencies become shorter as the trill picks up speed. The lines also start to fade and become lighter as the trill continues, indicating that the violist is doing a decrescendo, which is written in the score. At the end of this trill, there is a turn again, this time down to a C_5. The performer slows down for this turn as well, as shown by the long horizontal line at 523 Hz.
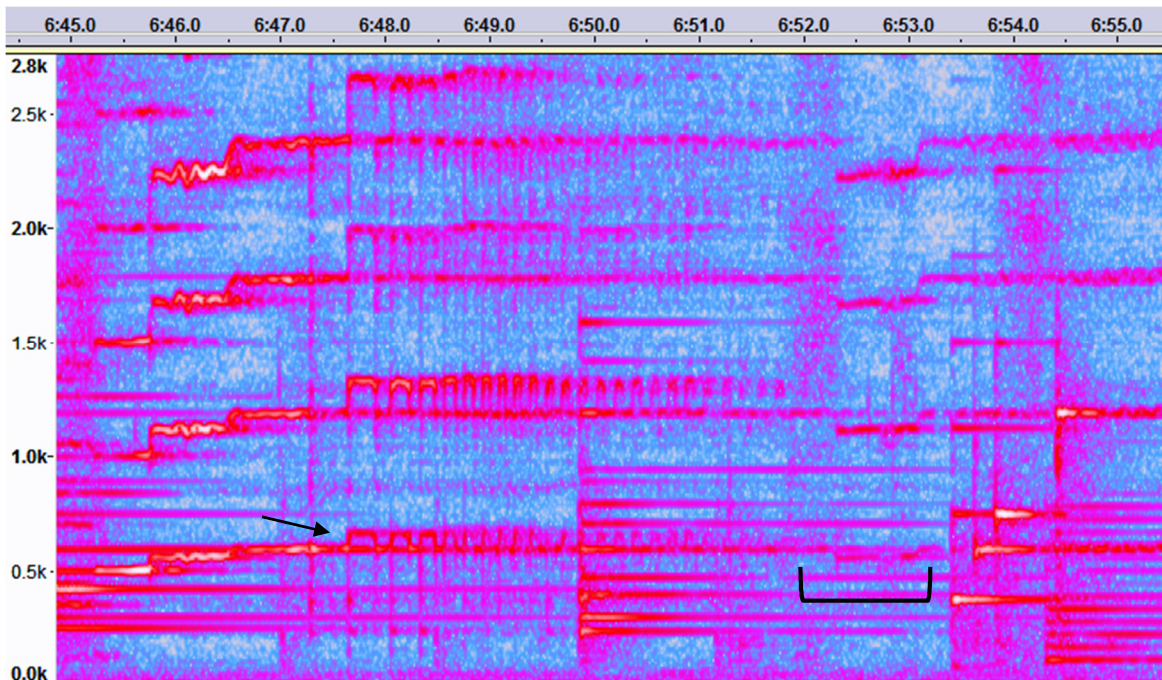
Figure 5.1: Spectrogram of measure 89. The arrow indicates the trill and the bracket indicates the turn.

Figure 5.2: Score excerpt of measures 88 and 89.

There is one non-musical component that occurs frequently throughout the piece that is worth mentioning since it appears in the spectrogram. This component is most likely the performer's breathing. The audio recording equipment used must have been very sensitive or placed too close to the performer's face, which produced the excess noise found in the spectrogram in Figure 6. It is less likely that this noise was produced by the viola since it is not indicative of any sounds a viola would normally produce. This noise is represented by vertical, purple lines in the spectrogram since it does not identify with any certain pitch.
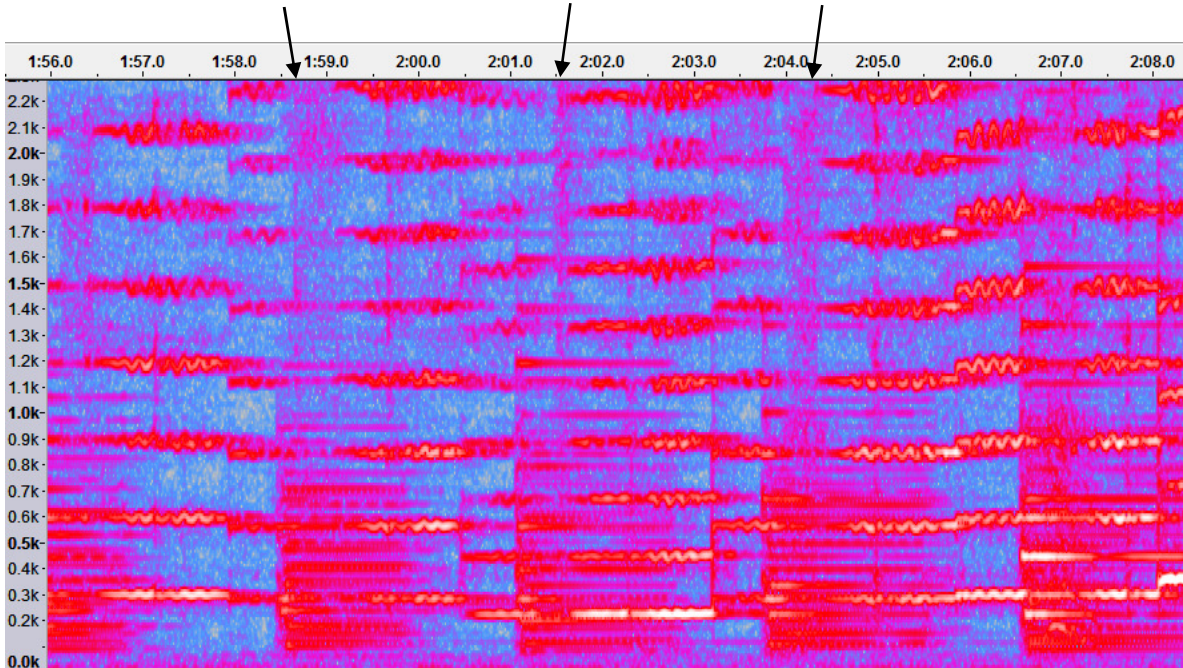
Figure 6: Spectrogram showing the representation of the performer's breathing within the piece.

Moving away from spectrograms, the score can be used alone to analyze chords and chord progressions. The first half of the piece is in d-minor because the main accidental used is C♯, which is the leading tone to D. This, along with the key signature of one flat and the main melody beginning and ending on D, makes it d-minor. Since the second half is then in D-major, there has to be a point where these two keys meet and switch. This occurs in measure 58, or 4:30 in the recording. The third beat of this measure contains an eighth note chord in the piano which has the notes A, C♯, E, and G in various octaves. These notes make up the A-major seventh chord, which is the major five chord in both d-minor and D-major. Since the following measure changes to a key signature of two sharps, this makes it a pivot point since it bridges the gap between the two keys seamlessly.

As for chord progressions, the one shown below comes from the first instance of the main melody, from measures 9-12. Each chord's location in the music is identified by the top row of the chart; "m" stands for measure and "b" stands for beat, so for example, "m9/b2" translates to "measure 9, beat 2." The chord progression in the second row uses the note names of the chords, while the third row uses the roman numerals based on the d-minor scale. This structure and abbreviations will be used for the remainder of the chord progressions analyzed.

m9/b2	m9/b4	m10/b2	m10/b4	m11/b2	m12/offbeat of b1	m12/b2	m12/offbeat of b2	m12/b3
Dm	A^7/E	A	A^7/G	Dm/F	Gm or E°/G	Dm/A	A^7	Dm
i	V^{43}	V	V^{42}	i^6	iv or ii^6	i^{64}	V^7	i

The sixth chord in this progression (m12/offbeat of b1) could be analyzed two ways: as a Gm or E°/G chord. If the E_4 in the viola part is ignored, then the chord is only made up of G's and a B♭, making it a G chord missing the third note of the triad. If the viola note is taken into account, the chord then contains G's, a B♭, and an E, which could be seen as a first inversion of an E° chord. The G chord is the more likely analysis because it follows the general rules for chord progressions that i goes to iv and iv goes to i. If E°/G is used, this breaks the rule that ii° cannot go to i. Despite this, it could have been the composer's intent to break this rule. Also, in later chord progressions of this melody, an E does appear in the piano part, forcing these later chords to be E°.

The next time this main melody appears is from measures 13-16, directly after the previous instance of this melody. This time, the viola part is shifted directly by one octave, though the piano part does not do the same. Instead, it contains almost the exact chord progression as previously (though the places of the individual notes within the chords do change). The only chords that change slightly are the first one, which switches from a i to a i^6, and the third one, which switches from V to V^7.

m13/b2	m13/b4	m14/b2	m14/b4	m15/b2	m16/offbeat of b1	m16/b2	m16/offbeat of b2	m16/b3
Dm/F	A^7/E	A^7	A^7/G	Dm/F	E°/G	Dm/A	A^7	Dm
i^6	V^{43}	V^7	V^{42}	i^6	ii^6	i^{64}	V^7	i

The third occurrence of this melody is from measures 25-28. This time, the first three chords are different while the rest remain the same. Instead of starting with the tonic chord, this progression starts with the VI chord, B♭. In some cases, using the VI chord from d-minor to start a progression could indicate a shift into the key of B♭-major, since the B♭ chord is diatonic to both keys. In this case, though, the music remains in d-minor as indicated by the use of C♯ in the next three chords.

m25/b2	m25/b4	m26/b2	m26/b4	m27/b2	m28/offbeat of b1	m28/b2	m28/offbeat of b2	m28/b3
B♭	C♯°/G	A^7/E	A^7/G	Dm/F	E°/G	Dm/A	A^7	Dm
VI	$vii°^{43}$	V^{43}	V^{42}	i^6	ii^6	i^{64}	V^7	i

The last occurrence of this melody is from measures 47-50. The chord progression here is very similar to the one previous to it at measures 25-28, but with a slight difference. The last chord is now a VI just like the beginning of the progression. The composer does this because the piano part goes right into more notes instead of sustaining the last chord, as was done with the previous three progressions. Also notice the right hand of the piano part is using tremolo

instead of metered chords as before. Because of this, all of the notes involved in each tremolo are considered when finding the chord progression.

m47/b2	m47/b4	m48/b2	m48/b4	m49/b2	m50/offbeat of b1	m50/b2	m50/offbeat of b2	m50/b3
B♭	C#°/G	A^7/E	A^7/G	Dm/F	E°/G	Dm/A	A^7	B♭
VI	vii°^{43}	V^{43}	V^{42}	i^6	ii^6	i^{64}	V^7	VI

In addition to chord progressions, there are various patterns in pitch and rhythm that can be analyzed with the use of the score. For instance, the viola line in measures 31-32 and measures 35-36 have the same motif, though the note values are changed. This change is a diatonic scale shift of T_3. The only notes that do not follow the scale shift of T_3 are in the third beat of measure 31, the A_4 and G_4. The two pitches shift up to E_5 and D_5, respectively, which indicate a diatonic scale shift of T_4.
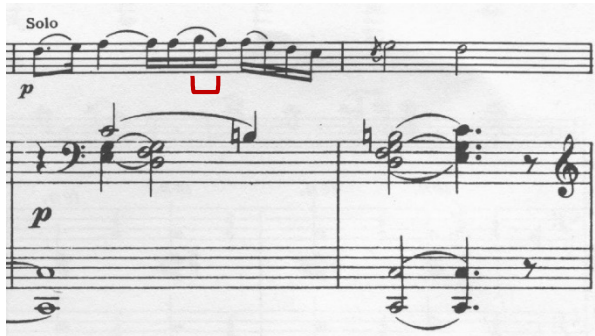

Figure 7.1: Score excerpt of measures 31-32. The bracket indicates the two pitches, A_4 and G_4, which follow T_4.

Figure 7.2: Score excerpt of measures 35-36. The bracket indicates the two pitches, E_5 and D_5, which follow T_4.

Though the piano part mostly consists of large chords and arpeggios, there are some instances where the piano line synchronizes with the viola line. One of these instances occurs in measure 17. Here the viola is playing a simple melody that consists of eight eighth notes in the following order: G_4 A_4 $B♭_4$ A_4 G_4 C_5 $B♭_4$ A_4. Inside the piano part, there is also a line, as indicated in Figure 8, which consists of eight eighth notes: $B♭_3$ C_4 D_4 C_4 $B♭_3$ $E♭_4$ D_4 C_4. When both of these parts are played together, intervals of 6ths are created, which creates a simple harmony to complement the melody in the viola. This same type of synchronization occurs in measure 19 as well, though in this case there are two lines in the piano part that synchronize with the viola part. The viola part here is exactly one octave below the part from measure 17, with the exception of the third beat where an extra note is added. In the piano part, one line of eighth notes occurs in the treble line and is exactly one octave above the viola part. The second line is in the bass line and is an interval of a 3rd above the viola part.

Figure 8: Score excerpt of measures 17-20. The red arrow indicates the eighth note line in the piano part that synchronizes with the viola part in measure 17. The blue arrows indicate the eighth note lines in the piano part in measure 19.

Though much of the rhythm in Kol Nidrei lacks pattern, some passages use cyclic rhythms, which are constructed using the Euclidean Algorithm to evenly space out note onsets.

The first passage that will be analyzed is part of the main melody in the viola from measures 9 and 10. These two measures have 11 note onsets out of a possible 16, but for the purpose of constructing a cyclic rhythm, the first note which is a quarter note will be treated as two eighth notes. This is allowable because the first half of the quarter note is played straight, while the second half is played with significant vibrato, giving the sense that it is two separate notes, as shown in Figure 9. This will then give 12 note onsets on a 16-hour clock, and all note onsets now represent eighth notes. The Euclidean Algorithm can now be used as shown below, where 1's represent note onsets and 0's represent lack of notes.

$$1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1,\ 0\ 0\ 0\ 0 \qquad (16 = 12 + 4)$$

$$1\ 0\ \ 1\ 0\ \ 1\ 0\ \ 1\ 0,\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \qquad (12 = 4 + 8)$$

$$1\ 0\ 1\ 1\ \ 1\ 0\ 1\ 1\ \ 1\ 0\ 1\ 1\ \ 1\ 0\ 1\ 1 \qquad (8 = 4 * 2 + 0)$$

The result is 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1, which corresponds to the clock hours 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16. By applying the shift T_1, the new clock hours are 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, which are the note onsets found in measures 9-10. This cyclic rhythm is also found in measures 13-14, 25-26, and 47-48. A clock of this rhythm and the excerpt from the score are shown below.
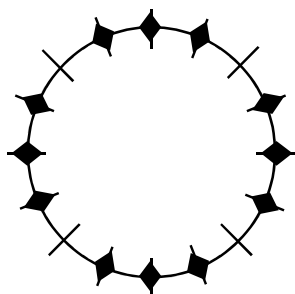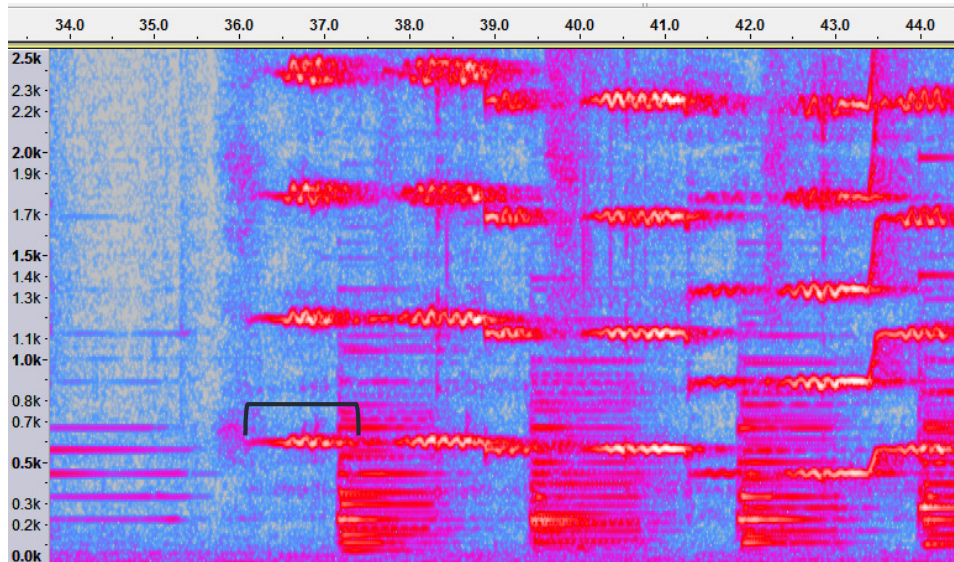
Figure 9: Spectrogram of measures 9-10. The bracket indicates the first note, which is treated as two eighth notes due to the vibrato.

Another rhythm in Kol Nidrei that can be constructed in the same fashion appears in measures 82, 84, and 86 in the viola line. Here, there are 14 note onsets out of a possible 16, so the Euclidean Algorithm is used again to produce a rhythm of 14 note onsets on a 16-hour clock.

$$1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1,\ 0\,0 \qquad (16 = 14 + 2)$$

$$1\,0\ 1\,0,\ 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 \qquad (14 = 2 + 12)$$

$$1\,0\,1\,1\,1\,1\,1\,1\ 1\,0\,1\,1\,1\,1\,1\,1 \qquad (12 = 2 * 6 + 0)$$

The result is 1 0 1 1 1 1 1 1 1 0 1 1 1 1 1 1, which corresponds to the clock hours 1, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16. By this time applying the shift T_{-1}, the new clock hours are 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, which are the note onsets in measures 82, 84, and 86. A clock of this rhythm is shown below.
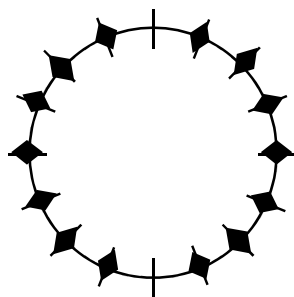
Overall, Kol Nidrei can be clearly analyzed through the use of spectrograms to create visuals, score analysis, and the Euclidean Algorithm. Despite the complexity and discontinuity of the music itself, there are still many mathematical and musical patterns lying within, and many that are not mentioned in this analysis.